266. Proposed by DR. O. E. GLENN, Drury College.

Given the feet of the three perpendiculars from any point a on the circumcircle to the sides of the triangle are collinear, then if on the three chords $\overline{ab}$, $\overline{ac}$, $\overline{ad}$, as diameters circles be described, the points of intersection of these circles are collinear. [Salmon's *Higher Plane Curves*].

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Let A, B, and C be the points of intersection of the circumferences described on ab, ac, and ad as diameters. Then since the diameters are concurrent at a, and A is common to two circles which have a common chord, $\angle cAa$ and $\angle dAa$ are right angles, and therefore their two non-coincident sides form a straight line through c and d. Likewise B is collinear with b and d, and consequently with c and d. Also C is collinear with b and c since there cannot be more than one perpendicular to the common chord at C. Therefore A, B, and C are collinear.

II. Solution by the PROPOSER.

By inversion with respect to a as a center, ab, ac, ad remain invariant. The given circle inverts into its radical axis with the circle of inversion, and the three circles in question into a triangle in the circle of inversion. The vertices of this triangle lie on a circle passing through a, for perpendiculars from a on the sides are collinear. Hence the invert of a circle through the intersections of the circles in question is a circle through a, and hence the circle through these intersections is a straight line.

Also solved by G. B. M. Zerr.

GROUP THEORY.

10. Proposed by O. E. GLENN, Ph. D.

Find the order of the group of isomorphisms (H) of the group (G) of order p^4 defined by the relations $P_1^{p^2}=P_2^{p^2}=I$, $P_1P_2=P_2P_1$.

Solution by the PROPOSER.

In the general isomorphism represented by

$$*J=\begin{pmatrix} P_1, & P_2 \\ P_1^xP_2^y, & P_1^zP_2^w \end{pmatrix}$$

the operation $P_1^{\alpha}P_2^{\beta}$ corresponds to $P_1^{x'}P_2^{y'}$, where

$$\left.\begin{matrix} x'\equiv \alpha x+\beta z \\ y'\equiv \alpha y+\beta w \end{matrix}\right. \pmod{p^2} \text{.........(I).}$$

Hence H is isomorphic with the congruence group defined by (I) and its order h equals the number of sets of values x, y, z, w for which

*Burnside, *Finite Groups*, p. 222.

$$xw-zy\equiv S(\bmod p^2)\,;\ (S \text{ not}\equiv 0)$$

provided only that at least one letter of the set (x, y), and one of (z, w) are prime to p. Consider first the case $zy\equiv 0 \pmod{p^2}$. This may happen when

1) $z\equiv 0$, with $y\equiv \Phi(p^2)=p(p-1)$ values.
2) $z\equiv 0$, with $y\equiv p^2-1-\Phi(p^2)=p-1$ values.
3) $z\equiv 0$, $y\equiv 0$.
4) $z\equiv \Phi(p^2)=p(p-1)$ values, with $y\equiv 0$.
5) $z\equiv p^2-1-\Phi(p^2)=p-1$ values, with $y\equiv 0$.
6) $z\equiv p^2-1-\Phi(p^2)=p-1$ values, with $y\equiv p^2-1-\Phi(p^2)\equiv p-1$ values.

Then for

1), x may assume p^2-1, and w, $p(p-1)$ values.
2), x may assume $p(p-1)$, and w, $p(p-1)$ values.
3), x may assume $p(p-1)$, and w, $p(p-1)$ values.
4), x may assume $p(p-1)$, and w, p^2-1 values.
5), x may assume $p(p-1)$, and w, $p(p-1)$ values.
6), x may assume $p(p-1)$, and w, $p(p-1)$ values.

Hence there are

$$2(p^2-1)p^2(p-1)^2+2p^2(p-1)^3+2p^2(p-1)^2$$
$$+2(p^2-1)p^2(p-1)^2+2p^2(p-1)^3+2p^2(p-1)^4 \ldots\ldots\ldots (\text{II})$$

sets of values with either $xw\equiv 0$ or $zy\equiv 0$, for which S is not$\equiv 0 \pmod{p^2}$.

Next let two of the parameters x, y, z, w, be divisible by p, without making $xw\equiv 0$ or $zy\equiv 0$. These must be (x and z) or (y and w). Let x and z be multiples of p, [$(p-1)^2$ sets of them]. Then to y may be assigned $\Phi(p^2)=p(p-1)$ values, for each one of which there are $^*dv(x;\, p^2)=p$ values of w which make $S\equiv 0$ and $p(p-1)-p=p^2-2p$ values giving a value of S not$\equiv 0$; and similarly, taking y and w, divisible by p. We thus obtain

$$2p^2(p-2)(p-1)^3 \text{ sets of } x,\ y,\ z,\ w,\ \ldots\ldots\ (\text{III}).$$

Next let it be assumed that one and but one parameter x, y, z, w, is divisible by p. Then for *all* the $\Phi(p^2)$ values each, of the remaining three parameters, S is not$\equiv 0$, and since there are four ways of taking one letter divisible by p, and $p-1$ values for that letter, this case gives

$$4p^3(p-1)^4 \text{ additional sets } \ldots\ldots\ (\text{IV}).$$

Finally, let all the parameters be prime to p. To each of three of them we may assign any one of $\Phi(p^2)=p(p-1)$ values, and then one value of the remaining one is determined making $S\equiv 0$, so that there are $p(p-1)-1$ values of

*Mathews, *Theory of Numbers*, p. 11.

this remaining one giving a value of S not$\equiv 0$ (mod p). Thus are obtained $p^3(p-1)^3(p^2-p-1)$ new sets ....(V).

This exhausts the cases to consider. The sum of II, III, IV, and V is h, the order of the automorph of G. This sum is

$$h=p^3(p-1)^2(p+1)(p^2+p-1).$$

MECHANICS.

184. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

A sphere, radius a, rests between two parallel thin perfectly rough rods A and B in the same horizontal plane at a distance apart equal to $2c$; the sphere is turned about A until its center is very nearly vertically over A; it is then allowed to fall back. Prove that it will rock between A and B if $10c^2<7a^2$; also, that θ_r, the angle through which it will turn after the rth impact is given by the equation $\cos\theta_r=\dfrac{\sqrt{(a^2-c^2)}}{a}+\dfrac{a-\sqrt{(a^2-c^2)}}{a}\left(1-\dfrac{10c^2}{7a^2}\right)^{2r}$.

I. Solution by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Let ω'=the angular velocity of the sphere about its center at any time t' from the beginning of the first stage of motion; k, k_1 the radii of gyration about the center and a point of the surface, respectively; m=its mass, ω_1, ω_2, etc., the angular velocities about the center just after the first, second, etc., impacts; V'=the linear velocity of the center just before the first impact, $p=(a^2-2c^2)/a$ =the perpendicular from a point of contact upon the direction of V', θ'=the angle the radius makes with the vertical corresponding to ω', and g=the acceleration of gravity. Then for the first impact, the moment of angular momentum is

$$mk_1^2\omega_1=(k^2\omega'+V'p)m\text{ ....(1), or, }\omega_1=\frac{k^2\omega'+V'p}{k_1^2}\text{ ....(2).}$$

For the first stage of motion,

$$mk_1^2\frac{d^2\theta'}{dt'}=mga\sin\theta'\text{ .... (3).}$$

Integrating (3), noticing that initially, $d\theta'/dt'=0$, $\cos\theta'=1$,

$$mk_1^2\frac{d\theta'^2}{dt'^2}=2mga(1-\cos\theta')\text{ ....(4).}$$

When the first stage of motion is completed,

$$\cos\theta'=\frac{\sqrt{(a^2-c^2)}}{a}\text{, and (4) then is, since }k_1^2=\tfrac{7}{5}a^2\text{ ....(5),}$$